Math 409 Midterm 1 practice #1

Name: ____________________

This exam has 3 questions, for a total of 100 points.

Please answer each question in the space provided. No aids are permitted.

Question	Points	Score
1	40	
2	30	
3	30	
Total:	100	

Question 1. (40 pts)

In each of the following eight cases, indicate whether the given statement is true or false. No justification is necessary.

(a) Let E be a set and suppose that there exists a surjective function $f : \mathbb{R} \to E$. Then E is uncountable.

Solution: False.

(b) If E is a subset of $\mathbb{R}$ which has a supremum, then the set $-E = \{-x : x \in E\}$ has an infimum.

Solution: True.

(c) Let $a \in \mathbb{R}$. Then $|a| < \varepsilon$ for all $\varepsilon > 0$ if and only if $a = 0$.

Solution: True.

(d) If $\{E_x\}_{x\in\mathbb{R}}$ is a collection of finite sets indexed by the real numbers, then $\bigcup_{x\in\mathbb{R}} E_x$ is at most countable.

Solution: False.

(e) Every subset of $\mathbb{R}$ has at most two suprema.

Solution: True.

(f) Let $f\colon \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$. Then $f^{-1}([0,1]) = [-1,1]$.

Solution: True.

(g) Let $A_1, A_2, A_3, \cdots$ be nonempty finite subsets of $\mathbb{N}$ such that $A_n \cap A_m = \emptyset$ for all distinct $n, m \in \mathbb{N}$. Define the function $f\colon \mathbb{N} \to \mathbb{N}$ by declaring $f(n)$ to be the least element of A_n. Then f is injective.

Solution: True.

(h) Let $A_1, A_2, A_3, \cdots$ be nonempty bounded subsets of $\mathbb{R}$ such that $A_n \cap A_m = \emptyset$ for all distinct $n, m \in \mathbb{N}$. Define the function $f\colon \mathbb{N} \to \mathbb{R}$ by $f(n) = \sup A_n$. Then f is injective.

Solution: False.

Question 2. (30 pts)

(a) State the well-ordering principle.

Solution: If E is a nonempty subset of $\mathbb{N}$, then E has a least element.

(b) Prove that $2^{n-1} \leq n!$ for all $n \in \mathbb{N}$.

Solution: Let $A(n)$ be the statement that

$$2^{n-1} \leq n!.$$

If $n = 1$, then the LHS (left hand side) is $2^0 = 1$ and the RHS (right hand side) is $1! = 1$. Therefore, $A(1)$ is true.
Now suppose $A(n)$ is true for some $n \geq 1$. In particular, $n + 1 \geq 2$. Then for $A(n+1)$,

$$\text{LHS} = 2^n = 2 \cdot 2^{n-1} \leq 2 \cdot n! \leq (n+1) \cdot n! = (n+1)! = \text{RHS}.$$

Thus $A(n+1)$ is true whenever $A(n)$ is true. We conclude by induction that $A(n)$ is true for all $n \in \mathbb{N}$.

Question 3. (30 pts)

(a) State the completeness axiom for $\mathbb{R}$.

Solution: For every nonempty subset $E \subset \mathbb{R}$, if E is bounded above, then E has a finite supremum.

(b) Let A be a nonempty bounded subset of $\mathbb{R}$, and consider the set $B = \{x^2 \colon x \in A\}$. Prove that $\sup B$ exists.

Solution: B is nonempty, since A is nonempty. Because A is bounded, there exists $M \geq 0$ such that $|x| \leq M$ for all $x \in A$. Then we have

$$x^2 = |x|^2 \leq M^2$$

for all $x \in A$. In other words, B is bounded above by M^2. Now by the completeness of $\mathbb{R}$, we conclude that B has a finite supremum.

(c) Give an example to show that the equality $\sup B = (\sup A)^2$ may fail in part (b).

Solution: For example, let $A = \{-4, 1\}$. Then $\sup A = 1$, hence $(\sup A)^2 = 1$. On the other hand, $B = \{16, 1\}$ and $\sup B = 16$. So $\sup B \neq (\sup A)^2$.